IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

Three-dimensional lattice gauge theory as self-avoiding random surfaces

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1987 J. Phys. A: Math. Gen. 20 L535

(http://iopscience.iop.org/0305-4470/20/8/009)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 129.252.86.83
The article was downloaded on 31/05/2010 at 10:44

Please note that terms and conditions apply.

J. Phys. A: Math. Gen. **20** (1987) L535-L538. Printed in the UK

LETTER TO THE EDITOR

Three-dimensional lattice gauge theory as self-avoiding random surfaces

Thomas A Larsson

Department of Theoretical Physics, Royal Institute of Technology, S-10044 Stockholm, Sweden

Received 16 February 1987

Abstract. We construct a three-dimensional generalisation of the brick lattice and show that the partition function for a suitably modified SO(n) or SU(n) lattice gauge theory can be written exactly as a sum over closed surfaces on this lattice. For Ising lattice gauge theory, we derive a formal expression for the partition function.

It is widely believed that a lattice gauge theory at the critical point is equivalent to a gas of random surfaces (Fröhlich 1985, Schrader 1985). A numerical simulation consistent with this conjecture has been carried out by Karowski (1986), who found Ising-like exponents within error bars. For the three-dimensional Ising lattice gauge theory, such constructions have been made by several authors (e.g. Fradkin *et al* 1980, Kavalov and Sedrakyan 1986). Generalising a trick by Domany *et al* (1981), we will show how to rewrite a peculiar SO(n) symmetric lattice gauge theory defined on a particular three-dimensional lattice as a surface gas. Our construction also works for other gauge groups.

The formulation of the O(n) spin model as a gas of self-avoiding random loops was only possible on the honeycomb lattice. The reason for this is that the coordination number of this lattice is three, while the vertices of all other lattices border at least four links. The construction of Domany *et al* (1981) was generalised to arbitrary lattices by Larsson (1985), but the result now became a gas of self-intersecting loops with complicated statistics, which is much more difficult to treat.

In three dimensions there is no lattice with coordination number three, but we can construct one where no edge is on the boundary of more than three faces. This is done as follows. Start with a two-dimensional (2D) brick lattice (BL), which is isomorphic to the honeycomb lattice. The full black lines of figure 1 form a 2D BL. Erect

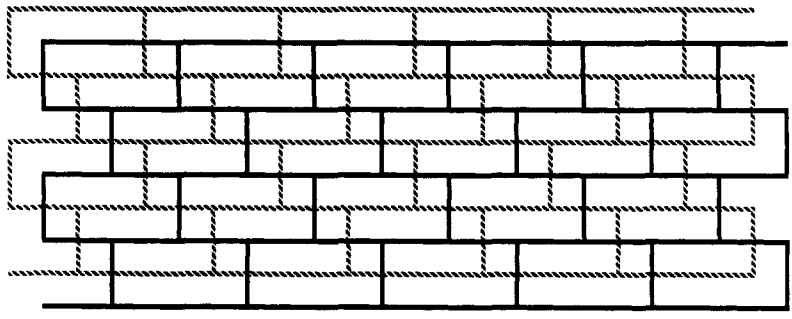

Figure 1. The three-dimensional brick lattice. One layer of bricks follows the lattice made of full lines, every other layer follows the lattice made of broken lines.

0305-4470/87/080535+04$02.50 © 1987 IOP Publishing Ltd

vertical links on each vertex of this lattice. Next place a new layer on top of the first, but displaced horizontally as shown in figure 1 (the grey lines). After this we repeat the procedure and add on more layers, every other layer displaced to avoid those lines on adjacent layers are precisely on top of each other. Put faces (plaquettes) wherever two bricks meet. The links and vertices are naturally defined as the borders between different plaquettes and links, respectively. The lattice so constructed we call the 3D BL and it has the following properties.

(i) Each brick is surrounded by 14 faces, six vertical ones and four on each of the two horizontal boundaries.

(ii) Each face borders two bricks.

(iii) Half of the horizontal faces are surrounded by four links, half are surrounded by six links.

(iv) One third of the vertical faces are surrounded by four links, two thirds are surrounded by six links.

(v) Each horizontal link borders three faces, two horizontal ones and one vertical.

(vi) Each vertical link borders three vertical faces, and no horizontal one.

(vii) Each vertex borders four links.

This construction can be generalised to any dimension by induction. Take each layer in the d-dimensional BL to be a $(d-1)$-dimensional BL and displace the layers horizontally so that $(d-2)$ cells of adjacent layers only intersect on $(d-3)$-dimensional sets. In particular, the 1D lattice with equal spacing is also the 1D BL. The d-dimensional BL obviously has the following properties.

(i) Each $(d-1)$-dimensional hyperface borders two d-dimensional bricks.

(ii) Each $(d-2)$-dimensional hyperlink borders three hyperfaces.

(iii) Each 1D link borders d faces.

(iv) Each 0D vertex borders $d+1$ links.

It is also obvious that there is no family of lattices, defined for all d, where the p cells border fewer $(p+1)$ cells.

Consider SO(n) lattice gauge theory, except in the case that $n=1$, where we choose the symmetry group to be O(1). This is called Ising lattice gauge theory (Wegner 1971, Kogut 1979). Normally one takes the Euclidean action to be, on the hypercubic lattice,

$$S = -\beta \sum_{\mathscr{P}} \operatorname{tr} O_i O_j O_k O_l \tag{1}$$

where $O_i = (O_i^{\alpha\beta})$ is an orthonormal n-dimensional matrix associated with the link i and $\langle ijkl \rangle$ are the links surrounding an elementary face (plaquette). The sum runs over all plaquettes $\mathscr{P}$ of the lattice, and the trace (tr) is with respect to the components of the product matrix, i.e.

$$\operatorname{tr} O_i O_j O_k O_l = O_i^{\alpha\beta} O_j^{\beta\gamma} O_k^{\gamma\delta} O_l^{\delta\alpha}.$$

The summation convention is used. However, we are normally only interested in the continuum limit of (1), where the lattice spacing goes to zero for fixed physical length scales. This is equivalent to letting the correlation length diverge for fixed lattice spacing, which is characteristic of a critical point. The critical behaviour is generally believed to be universal; it is not sensitive to the details of the model, nor of the particular lattice chosen, as long as the symmetry of the action and the dimensionality of the lattice is preserved. To study the continuum limit of the 3D version of (1), we can therefore modify the action to be

$$S = -\sum_{\mathscr{P}} \ln\left(1 + u \operatorname{tr} \prod_i O_i\right) \tag{2}$$

where the matrices now live on the links of the 3D BL and the sum runs over all faces of this lattice. ΠO_i is the product of matrices around a plaquette. This product contains four or six factors, depending on how many links border the plaquette.

There are two potential problems with this model. The lattice is necessarily anisotropic, in contrast to the 2D BL, which is isomorphic to the honeycomb lattice. Usually, spatial anisotropy does not alter the true critical behaviour, but it is only important in a crossover region (Aharony 1976). However, there are examples where it changes the asymptotic universality class of a model (Boyanovsky and Cardy 1982, Lawrie and Prudnikov 1984). We do not think that similar problems will arise here, but the possibility should be noted. Moreover, the model runs into trouble for $u > 1$, because then there are configurations with imaginary action. The same problem arises in the analogous two-dimensional spin models, without disturbing the continuum limit. In the Ising ($n = 1$) case, the actions (1) and (2) are identical, apart from an additive constant, if $u = \tanh \beta$.

The partition function is

$$Z(u) = \operatorname{Tr} e^{-S} = \operatorname{Tr} \prod_{\mathscr{P}} \left(1 + u \operatorname{tr} \prod_i O_i \right) \tag{3}$$

where the big trace (Tr) is with respect to all values of the fluctuating matrices $\{O_i\}$. If we normalise the group measure to unity, so that

$$\operatorname*{Tr}_{O_i} 1 = 1 \tag{4a}$$

we find

$$\operatorname*{Tr}_{O_i} O_i^{\alpha\beta} = \operatorname*{Tr}_{O_i} O_i^{\alpha\beta} O_i^{\gamma\delta} O_i^{\varepsilon\varphi} = 0 \tag{4b}$$

while

$$\operatorname*{Tr}_{O_i} O_i^{\alpha\beta} O_i^{\gamma\delta} = (1/n)(\delta^{\alpha\gamma}\delta^{\beta\delta} + \delta^{\alpha\delta}\delta^{\beta\gamma} - \delta^{\alpha\beta}\delta^{\gamma\delta}). \tag{4c}$$

The last relation follows because it is the only rank-four tensor which satisfies the orthogonality conditions: $O_i^{\alpha\beta} O_i^{\gamma\beta} = \delta^{\alpha\gamma}$ and $O_i^{\alpha\beta} O_i^{\alpha\gamma} = \delta^{\beta\gamma}$.

We are now in the position to expand out the product in (3). We consider each plaquette on the lattice to be occupied if we choose the second term for that plaquette, and otherwise we consider it empty. When we perform the trace (Tr), only those configurations will survive which consist of closed surfaces, because a boundary would imply that an odd number of faces next to that link were occupied, and that is impossible because of (4*b*). Moreover, these surfaces are self-avoiding, simply because there is no edge on the 3D BL where four faces can meet. Thus, the partition function (3) can be expressed as a sum over all configurations consisting of closed surfaces only.

Unfortunately, we have not been able to derive a formal expression for the partition function, analogous to the result of Domany *et al*, for general n. In the Ising case, however, the tensor (4*c*) is a scalar and the partition function becomes

$$Z(u) = \sum_{\mathscr{G}} \prod_s u \cdot u \cdot \ldots \cdot u = \sum_{\mathscr{G}} \prod_s u^{p_s} = \sum_{\mathscr{G}} u^p. \tag{5}$$

Here the sum runs over all configurations of closed surfaces $\mathscr{G}$ and the product runs over the s disjoint surfaces in the configuration $\mathscr{G}$. $p = \Sigma p_s$ is the total number of occupied plaquettes. Thus, the Ising lattice gauge theory is equivalent to a gas of self-avoiding closed surfaces with unit chemical potential for surfaces, whose partition

function for general chemical potential μ is $Z = \Sigma\, \mu^s u^p$, where s is the number of disjoint surfaces in a configuration. This result is completely analogous to the situation in two dimensions, where the chemical potential for loops is one in the Ising case.

In conclusion, we have generalised the result of Domany *et al*, who found that the $O(n)$ spin model on the 2D BL can be written as a theory of closed loops, to the 3D $SO(n)$ lattice gauge theory, but we have not found an explicit formal expression for the partition function except when $n = 1$. This result can immediately be generalised to other groups, such as $SU(n)$. As long as the product of an odd number of matrices vanishes, a pure lattice gauge theory defined on the 3D BL, whose action is analogous to (2), is equivalent to some gas of self-avoiding closed surfaces. It might also be interesting to use actions like (2) in strong coupling expansions on four-dimensional lattices, because terms in the series could be calculated for all n at once.

The generalisation to higher dimensions is not straightforward. In four dimensions, one could imagine variables living on the faces of the 4D BL and take the action to be a sum over 3D hyperfaces. However, it is not obvious how to define multiplication of non-commuting entities around a 3 cell. The Ising case can always be defined, because there the product of spins can be performed in any order. Using exactly the same line of reasoning as above, we find that the partition function of a model of Ising spins living on the $(d-2)$ cells and interacting around the $(d-1)$ cells of a d-dimensional BL can be written as a sum over closed $(d-1)$-dimensional hypersurfaces. Formally, the result is exactly the same as (5), but where p now stands for the number of hyperfaces.

References

Aharony A 1976 *Phase Transitions and Critical Phenomena* vol 6 ed C Domb and M S Green (New York: Academic) p 358

Boyanovsky D and Cardy J L 1982 *Phys. Rev.* B **26** 154

Domany E, Mukamel D, Nienhuis B and Schwimmer A 1981 *Nucl. Phys.* B **190** [FS3] 279

Fradkin E, Srednicki M and Susskind L 1980 *Phys. Rev.* D **21** 2885

Fröhlich J 1985 *Progress in Gauge Field Theory* ed G't Hooft *et al NATO Advanced Study Inst. Ser. B* no 115 (New York: Plenum)

Karowski M 1986 *J. Phys. A: Math. Gen.* **19** 3375

Kavalov A R and Sedrakyan A G 1986 *Phys. Lett.* **173B** 449

Kogut J B 1979 *Rev. Mod. Phys.* **51** 659

Larsson T A 1985 *Phys. Rev.* B **31** 4710

Lawrie I D and Prudnikov V V 1984 *J. Phys. C: Solid State Phys.* **17** 1655

Schrader R 1985 *Commun. Math. Phys.* **102** 31

Wegner F 1971 *J. Math. Phys.* **12** 2259